

REGULARITY THEORY FOR THE MÖBIUS ENERGY

PHILIPP REITER

Institut für Mathematik
RWTH Aachen University
Templergraben 55
52062 Aachen, Germany

ABSTRACT. The Möbius energy, defined 1991 by O'HARA, is the most prominent example of a knot energy. In this text we will focus on the regularity of local minimizers (within a prescribed knot class) whose arc-length parametrization was shown to be $C^{1,1}$ by FREEDMAN, HE, and WANG. Later on, HE improved this result to C^∞ regularity. In this text we will briefly outline the main ideas of these two steps which require completely different approaches involving techniques from geometry and analysis. Moreover we explain how to rigorously derive the first variation of the Möbius energy and fix a gap in HE's treatise.

1. Introduction. Imagine the motion of a knotted charged fiber within a viscous liquid. Will it reach a stationary point minimizing its electrostatic energy? If so, will the resulting shape help to determine its knot type?

These questions led to the definition of a first example of a knot energy by FUKUHARA [4] in 1988. His idea was to find a “nicer shape” for a given knot in the same knot class, i. e. a representative that is as little entangled as possible with preferably large distances between different strands.

While FUKUHARA treated only piecewise linear curves, i. e. polygons, we can also claim smoothness as a second criterion for “nicely shaped” curves.

Knots and knot energies. In the following, a *knot* will denote an embedded closed curve in $C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^3)$. Two knots are said to belong to the same *knot class* if one can continuously be deformed into the other without self-intersections or pulling-tight of small knotted arcs, see Figure 1.

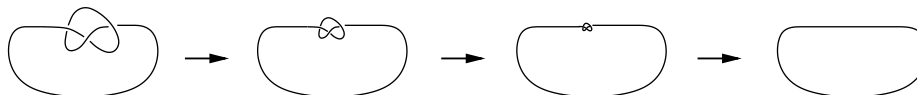


FIGURE 1. Pulling-tight of a small knotted arc

The fundamental paradigm when defining knot energies is to model *self-avoidance*, i. e. a knot energy has to blow up on sequences of embedded curves converging to a curve with a self-intersection. By this one hopes to avoid entangledness (and maybe non-smoothness) and, even more important, not to run into the danger of leaving

2000 *Mathematics Subject Classification.* Primary: 57M25, 58E10; Secondary: 53A04, 42A45.
Key words and phrases. Knot energy, bilinear Fourier multiplier, Möbius Energy, regularity.

the knot class during the minimizing process. But as we will see, self-avoidance alone does not prevent any of these three problems.

The definition of knot energies is also an attempt to give a new approach to the main problem of knot theory, to decide whether two knots belong to the same knot class. Furthermore, one hopes for future applications in physics and biochemistry.

Defining a knot energy. The first energy on smooth curves goes back to O’HARA [8]. His idea was to penalize pairs of points $(\gamma(s), \gamma(t))$ on a curve $\gamma : \mathbb{R}/(\ell\mathbb{Z}) \rightarrow \mathbb{R}^3$ having a small Euclidean distance $|\gamma(s) - \gamma(t)|$. This effect has to be regularized with respect to neighboring points which by their nature have a small Euclidean distance. To this end, he introduced the intrinsic distance $D_\gamma(s, t)$, see Figure 2.

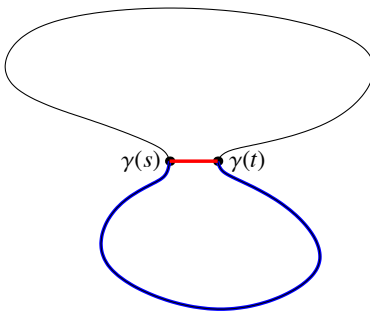


FIGURE 2. Euclidean (straight line) vs. intrinsic distance (lower arc)

In order to produce a sufficient singularity we have to square the respective terms obtaining $\frac{1}{|\gamma(s) - \gamma(t)|^2} - \frac{1}{D_\gamma(s, t)^2}$. Passing to the average by integration gives O’HARA’s definition

$$E(\gamma) := \iint_{(\mathbb{R}/(\ell\mathbb{Z}))^2} \left(\frac{1}{|\gamma(s) - \gamma(t)|^2} - \frac{1}{D_\gamma(s, t)^2} \right) |\dot{\gamma}(s)| |\dot{\gamma}(t)| ds dt,$$

where the factors $|\dot{\gamma}(s)| |\dot{\gamma}(t)|$ guarantee invariance under reparametrization when restricting to the class \mathcal{C} of absolutely continuous (i. e. $\gamma \in H^{1,1}$) injective curves with $\dot{\gamma} \neq 0$ a. e. By this property we can always assume arc-length parametrization $|\dot{\gamma}| \equiv 1$ which leads to $\gamma \in C^{0,1}$.

Since in the definition of E we only deal with (non-negative) distances, E is invariant under translations and orthogonal transformations. Moreover it does not depend on the orientation of the curve and is invariant under rescaling due to the factors $|\dot{\gamma}(s)| |\dot{\gamma}(t)|$. Thus we may assume $\ell = 2\pi$ without loss of generality.

Any finite-energy curve is bi-Lipschitz continuous with a bi-Lipschitz constant only depending on its energy, see O’HARA [8, Thm. 1.8]. This immediately proves self-avoidance.

Furthermore, E is continuous on injective $\gamma \in H^{2,2}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$ with $|\dot{\gamma}| \geq c > 0$, see [6, 10].

Möbius invariance and its consequences. The fundamental result of FREEDMAN, HE, and WANG [3] was the discovery of a somewhat “hidden” invariance — E is invariant under inversions on spheres, up to an additive constant if the curve passes through the center of the sphere. Altogether, E is invariant under Möbius

transformations in \mathbb{R}^3 , the group generated by inversions on planes and spheres. By virtue of this fact they coined the name *Möbius energy*.

While the Möbius invariance plays a fundamental role in almost all arguments in [3] there is one important drawback regarding the concept of knot energies: It is fairly simple to construct a sequence of curves featuring a pulling-tight as shown in Figure 1 while their energy is constant. Despite of its self-avoidance, minimizing the Möbius energy always provides the danger of leaving the knot class, not by self-intersection but by pulling-tight.

In this context, we would like to ask to what extent E does meet the intention of a knot energy. Of course, the Möbius energy is some measure for “entangledness” since any finite-energy curve is bi-Lipschitz continuous with a Lipschitz constant monotonously depending on its energy. On the other hand, applying a suitable Möbius transformation maps a given curve to a new curve of the same energy where its actual “knotting” is contained in a ball of arbitrary small radius. This does not fit in the general idea of a “nicely shaped” knot, so the Möbius energy is not a perfect measure in this sense.

Regularity properties of finite-energy curves can be analyzed locally, so that we can assume an arbitrarily small amount of energy by restricting to sufficiently small subarcs. By the fact that the bi-Lipschitz constant tends to 1 as $E \rightarrow 0$, this excludes “corners” and more so “cusps”, but does not imply differentiability. Even worse, by modifying a curve in a small neighborhood of a given point x we obtain a new curve that is not differentiable at x with energy arbitrarily close to that of the original curve, cf. [2]. In this respect, the Möbius energy is a “bad” measure for smoothness.

On the other hand, we obtain almost the best possible regularity when restricting to local minimizers of E (which exist at least in prime knot classes), see Theorem 5.

In spite of these drawbacks, E remains an interesting functional due to its geometric properties. Examples of knot energies which penalize entangledness and pulling tight while providing $C^{1,\beta}$ regularity are given by the family of $E^{(\alpha)}$ energies, see (1) below, and the Thickness, see GONZALEZ et. al. [5].

Minimizers of E . Since line segments are unique minimizers among all (open) curves and a line is mapped to a circle by inversion on a (suitable) sphere, FREEDMAN, HE, and WANG obtained as a first consequence of the Möbius invariance the minimizing property of circles among all closed curves [3, Cor. 2.2].

Since they were able to prevent only *one* knotted arc of a minimal sequence from pulling tight, they could prove existence of E -minimizers only within prime knot classes [3, Thm. 4.3]. A knot is said to be a *prime knot* if it cannot be decomposed into two non-trivial knots. It is an open question whether the minimizers are unique up to Möbius transformations, see KUSNER and SULLIVAN [7].

By the fact that circles are global minimizers, one may hope for some regularity of local minimizers. Here, we are no longer restricted to prime knots.

Theorem 1 ([3, Thm. 5.4]). *A local minimizer $\gamma \in C^{0,1}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$ within a prescribed knot class, $E(\gamma) < \infty$, parametrized by arc-length, has a Lipschitz continuous tangent, i. e. $\gamma \in C^{1,1}$.*

We will discuss this result in Section 2.

Moreover, FREEDMAN, HE, and WANG stated a formula for the first variation of E , cf. [3, Lem. 6.1]. While they only formally differentiate the integrand of

$E(\gamma + \tau h)$ with respect to $\tau = 0$, interchanging differentiation, integration and the limit process requires rather involved arguments which we sketch in Theorem 6.

Later on, Z.-X. HE [6] once more investigated the gradient of the Möbius energy. His topics are short-time existence of the gradient flow¹ associated to E and C^∞ -regularity of critical points by considering the Euler-Lagrange equation of E and applying a bootstrap argument involving pseudodifferential calculus.

Theorem 2 ([6, Cor. 5.3]). *Any local minimizer $\gamma \in C^{0,1}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$ within a prescribed knot class, $E(\gamma) < \infty$, parametrized by arc-length, is C^∞ smooth.*

Unfortunately, his arguments contain some major gaps. In Section 3 we sketch how to set up a rigorous proof involving the study of bilinear Fourier multipliers defined on fractional Sobolev spaces.

Our techniques from Section 3 also apply to the energy family

$$E^{(\alpha)}(\gamma) = \iint_{(\mathbb{R}/(2\pi\mathbb{Z}))^2} \left(\frac{1}{|\gamma(s) - \gamma(t)|^\alpha} - \frac{1}{D_\gamma(s, t)^\alpha} \right) |\dot{\gamma}(s)| |\dot{\gamma}(t)| \, ds \, dt, \quad (1)$$

where $\alpha \in [2, 3)$, see [10].

2. Local minimizers. In this section we will briefly sketch the proof of Theorem 1.

We start with a precise definition of local minimizers. A curve $\gamma \in \mathcal{C}$ is said to be a *local minimizer* if there is an open neighborhood U of image γ such that $E(\gamma) \leq E(\bar{\gamma})$ for any $\bar{\gamma} \in \mathcal{C}$ whose image is contained in U and which belongs to the same knot class as γ .

The most crucial argument is the following *reflection principle*. Let H be some open half-space in \mathbb{R}^3 , i. e. ∂H is a plane. Suppose that the image of $\gamma \in \mathcal{C}$, $E(\gamma) < \infty$, does not contain any point of H , but there are at least two disjoint points belonging to image $\gamma \cap \partial H$. They split up the curve into two subarcs γ_1, γ_2 where γ_1 is meant to be the shorter one. Briefly, “ $\gamma = \gamma_1 \cup \gamma_2$ ”.

Now we denote the reflection of γ_1 on ∂H by $\tilde{\gamma}_1$ and define a new curve γ_H in the obvious way by “ $\gamma_H := \tilde{\gamma}_1 \cup \gamma_2$ ”. Without loss of generality, we may assume arc-length parametrization for γ which also leads to an arc-length parametrized γ_H . Now, by $|\gamma(s) - \gamma(t)| \leq |\gamma_H(s) - \gamma_H(t)|$ for all $s, t \in \mathbb{R}/2\pi\mathbb{Z}$ and $D_\gamma \equiv D_{\gamma_H}$, we arrive at $E(\gamma) \geq E(\gamma_H)$. Moreover, $E(\gamma) = E(\gamma_H)$ holds if and only if the image of γ_1 or γ_2 is entirely contained in ∂H . To this end, we derive from $E(\gamma) = E(\gamma_H)$ and $D_\gamma \equiv D_{\gamma_H}$ the identity

$$\iint_{(\mathbb{R}/2\pi\mathbb{Z})^2} \left(\frac{1}{|\gamma(s) - \gamma(t)|^2} - \frac{1}{|\gamma_H(s) - \gamma_H(t)|^2} \right) \, ds \, dt = 0.$$

Since the integrand is non-negative it vanishes almost everywhere on $(\mathbb{R}/2\pi\mathbb{Z})^2$. By continuity we arrive at $|\gamma(s) - \gamma(t)| = |\gamma_H(s) - \gamma_H(t)|$ for all $s, t \in \mathbb{R}/2\pi\mathbb{Z}$. Assuming $\gamma(t) = \gamma_2(t) \notin \bar{H}$, this yields $|\gamma_1(s) - \gamma_2(t)| = |\tilde{\gamma}_1(s) - \gamma_2(t)|$ while $\gamma_1(s) \in \mathbb{R}^3 \setminus H$ and $\tilde{\gamma}_1(s) \in \bar{H}$. An elementary geometric argument gives $\gamma_1(s) = \tilde{\gamma}_1(s) \in \partial H$ as proposed. Of course, the converse is just a consequence of invariance under Euclidean transformations.

Since a half-space is mapped to a ball by a suitable Möbius transformation, we obtain the following statement by the Möbius invariance of E .

¹Very recently, a long-time existence result for the gradient flow of the Möbius energy has been established by BLATT [1].

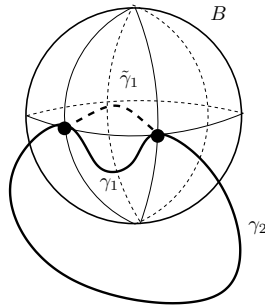


FIGURE 3.

Proposition 3 (Reflection principle). *Let B an (open) ball and $\gamma \in \mathcal{C}$, $E(\gamma) < \infty$, with image $\gamma \cap B = \emptyset$ and image $\gamma \cap \partial B$ containing at least two points. By decomposing “ $\gamma = \gamma_1 \cup \gamma_2$ ” as indicated in Figure 3 and defining “ $\gamma_B = \tilde{\gamma}_1 \cup \gamma_2$ ” where $\tilde{\gamma}_1$ denotes the reflection of γ_1 on ∂B , we obtain $E(\gamma) \geq E(\gamma_B)$ with equality if and only if the image of γ_1 or γ_2 is entirely contained in ∂B .*

This statement is the basic tool for the proof of

Lemma 4. *Let $\gamma \in \mathcal{C}$, $E(\gamma) < \infty$, be a local minimizer. Then there is a $\delta = \delta(\gamma) > 0$, such that for any ball B of radius $\leq \delta$ with image $\gamma \cap B = \emptyset$ the “contact set” image $\gamma \cap \partial B$ is connected (or empty).*

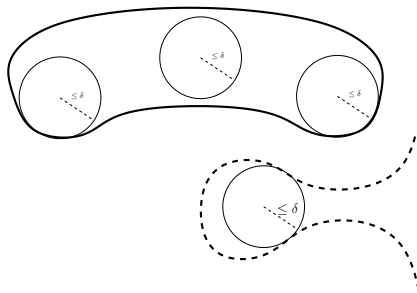


FIGURE 4. All balls of radius $\leq \delta$ that touch the dark line curve above produce a connected contact set which is not true for the dashed line curve.

Sketch of proof. By the definition of local minimizers there is some $\delta = \delta(\gamma) > 0$ such that $\|\gamma - h\|_{C^0} \leq 2\delta$ implies image $h \in U$ for any $h \in H^{1,1}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$. Assume $x, y \in \text{image } \gamma \cap \partial B$, $x \neq y$. We will show that there is an arc of γ that lies on ∂B and joins x and y . Since we are in the situation of Figure 3 we may construct the curve γ_B satisfying $E(\gamma) \geq E(\gamma_B)$. By construction, image $\gamma_B \subset U$. It is convincing that γ_B belongs to the same knot class as γ . (A rigorous proof of this fact is involving although using elementary arguments and requires to lessen δ , cf. [3, Sect. 5].) By the local minimizing property, we obtain $E(\gamma) \geq E(\gamma_B) \geq E(\gamma)$. Now Proposition 3 yields the desired arc on ∂B . \square

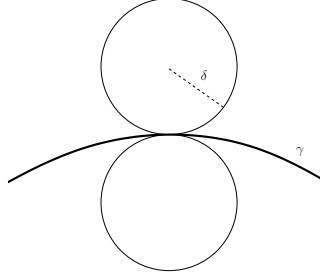


FIGURE 5. Cross section of the “horn torus” at some point of the curve γ

A consequence of Lemma 4 is the existence of a “horn torus” of radius δ around each point of image γ .

This implies the existence of a unique tangent at each point and furthermore provides a Lipschitz bound upon the tangents. This proves Theorem 1.

3. Critical points. A curve γ is said to be a *critical point* of E if its first variation $\delta E(\gamma, h) := \lim_{\tau \rightarrow 0} \frac{E(\gamma + \tau h) - E(\gamma)}{\tau}$ vanishes for any $h \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$.

Theorem 5 ([6, Thm. 5.1]). *Any E -critical point $\gamma \in H^{2,3}$ being injective and parametrized by arc-length is C^∞ smooth.*

The task of this section is to prove the preceding statement as it immediately gives the

Proof of Theorem 2. By Theorem 1, any local minimizer $\gamma : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^3$ belongs to $C^{1,1}$ when parametrized by arc-length. Moreover, there is some $\varepsilon = \varepsilon(\gamma) > 0$ such that all $h \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$ satisfying $\|\dot{\gamma} - \dot{h}\|_{C^0} \leq \varepsilon$ belong to the same knot class as γ , see [9] and recall the invariance of translation. Now $E(\gamma + \tau h) - E(\gamma) \geq 0$ for all $|\tau| \leq \varepsilon \|\dot{h}\|_{C^0}^{-1}$. This implies $\frac{E(\gamma + \tau h) - E(\gamma)}{\tau} \geq 0$ for $\tau > 0$ and $\frac{E(\gamma + \tau h) - E(\gamma)}{\tau} \leq 0$ for $\tau < 0$. Passing to the limit $\tau \rightarrow 0$ yields $\delta E(\gamma, h) = 0$, i. e. γ is E -critical. Since $C^{1,1}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3) = H^{2,\infty}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3) \subset H^{2,3}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$ we may apply Theorem 5. \square

When trying to compute the first variation of E one faces the problem if it is justified to interchange differentiation and the double integral.

$$\begin{aligned} \delta E(\gamma, h) &= \left. \frac{d}{d\tau} \right|_{\tau=0} E(\gamma + \tau h) \\ &= \left. \frac{d}{d\tau} \right|_{\tau=0} \iint_{(\mathbb{R}/(2\pi\mathbb{Z}))^2} \left(\frac{1}{|(\gamma + \tau h)(s) - (\gamma + \tau h)(t)|^2} - \frac{1}{D_{\gamma + \tau h}(s, t)^2} \right) |(\dot{\gamma} + \tau \dot{h})(s)| |(\dot{\gamma} + \tau \dot{h})(t)| \, ds \, dt \\ &\stackrel{?}{=} \iint_{(\mathbb{R}/(2\pi\mathbb{Z}))^2} \left. \frac{d}{d\tau} \right|_{\tau=0} \left[\left(\frac{1}{|(\gamma + \tau h)(s) - (\gamma + \tau h)(t)|^2} - \frac{1}{D_{\gamma + \tau h}(s, t)^2} \right) |(\dot{\gamma} + \tau \dot{h})(s)| |(\dot{\gamma} + \tau \dot{h})(t)| \right] \, ds \, dt \end{aligned}$$

A formula for the latter term is straightforward and already appeared in [3, Lem. 6.1]. FREEDMAN, HE, and WANG showed that this term does in fact exist

under certain assumptions while the integral has to be understood as a *principle value*, i. e.

$$\lim_{\varepsilon \searrow 0} \iint_{|s-t| \geq \varepsilon} \frac{d}{d\tau} \Big|_{\tau=0} \left[\left(\frac{1}{|(\gamma+\tau h)(s) - (\gamma+\tau h)(t)|^2} - \frac{1}{D_{\gamma+\tau h}(s,t)^2} \right) |(\dot{\gamma}+\tau \dot{h})(s)| |(\dot{\gamma}+\tau \dot{h})(t)| \right] ds dt. \quad (2)$$

Neither [3] nor [6] contain a proof for the admissibility of commutating differentiation, integration, and the limit process, so we will briefly sketch the argument.

Theorem 6. *Let $\gamma, h \in H^{2,2}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$, γ injective and $|\dot{\gamma}| \geq c > 0$. Then the first variation of E at γ in direction h exists and is given by (2).*

Proof. It is convenient to start with an approximate functional, namely

$$E_\varepsilon(\gamma) := \iint_{|s-t| \geq \varepsilon} \underbrace{\left(\frac{1}{|\gamma(t+w) - \gamma(t)|^2} - \frac{1}{D_\gamma(t, t+w)^2} \right) |\dot{\gamma}(t+w)| |\dot{\gamma}(t)|}_{=: f(\gamma; t, w)} dw dt,$$

which can be differentiated just via LEBESGUE's theorem. Its first variation

$$I_\varepsilon(\gamma, h) := \iint_{|s-t| \geq \varepsilon} \underbrace{\frac{d}{d\tau} \Big|_{\tau=0} f(\gamma + \tau h; t, w)}_{=: g(\gamma, h; t, w)} dw dt$$

turns out to be a continuous mapping

$$\tau \mapsto I_\varepsilon(\gamma + \tau h, h)$$

for $|\tau| \ll 1$ and $\varepsilon > 0$. The technical part is now to show that

$$\tau \mapsto \lim_{\varepsilon \searrow 0} I_\varepsilon(\gamma + \tau h, h)$$

is in fact also continuous for $|\tau| \ll 1$. It essentially relies on Taylor expansions which demand $\gamma, h \in C^3$ (or at least $C^{2,1}$). Now we use FUBINI's theorem to obtain

$$\begin{aligned} \frac{E_\varepsilon(\gamma + \tau h) - E_\varepsilon(\gamma)}{\tau} &= \iint_{|s-t| \geq \varepsilon} \frac{f(\gamma + \tau h; t, w) - f(\gamma; t, w)}{\tau} dw dt \\ &= \iint_{|s-t| \geq \varepsilon} \int_0^1 g(\gamma + \vartheta \tau h, h; t, w) d\vartheta dw dt = \int_0^1 \iint_{|s-t| \geq \varepsilon} g(\gamma + \vartheta \tau h, h; t, w) dw dt d\vartheta \\ &= \int_0^1 I_\varepsilon(\gamma + \vartheta \tau h, h) d\vartheta. \end{aligned}$$

Applying LEBESGUE's theorem we may first pass to the limit $\varepsilon \searrow 0$ and then send $\tau \rightarrow 0$ which reveals the existence of the first variation $\delta E^{(\alpha)}(\gamma, h) = I(\gamma, h) := \lim_{\varepsilon \searrow 0} I_\varepsilon(\gamma, h)$.

Since I continuously extends to $\gamma, h \in H^2$ if γ is parametrized by arc-length, cf. [6, Lem. 4.2], and E is continuous on H^2 curves, we may transfer this result to arbitrary regular H^2 curves via some reparametrization argument that essentially relies on LUSIN's theorem. The details are to be found in [10, Chap. 1]. \square

Computing the first variation of the Möbius energy according to Theorem 5 we arrive at

$$\delta E(\gamma, h) = 2 \lim_{\varepsilon \searrow 0} \iint_{|s-t| \geq \varepsilon} \left(\left\langle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}, \dot{h}(t) \right\rangle - \frac{\langle \gamma(s) - \gamma(t), h(s) - h(t) \rangle}{|\gamma(s) - \gamma(t)|^2} \right) \frac{|\dot{\gamma}(s)| |\dot{\gamma}(t)|}{|\gamma(s) - \gamma(t)|^2} ds dt.$$

Interestingly, it does neither involve D_γ nor its derivative which is compatible with [3, Lem. 6.1].

It is convenient to introduce a “linearization” Q derived from δE by assuming arc-length parametrization ($|\dot{\gamma}| \equiv 1$) which also implies $|\gamma(s) - \gamma(t)|^2 = (s - t)^2 + \mathcal{O}(|s - t|^4)$. Let

$$Q(\gamma, h) := \lim_{\varepsilon \searrow 0} \iint_{|s-t| \geq \varepsilon} \left(\left\langle \dot{\gamma}(t), \dot{h}(t) \right\rangle - \frac{\langle \gamma(s) - \gamma(t), h(s) - h(t) \rangle}{(s - t)^2} \right) \frac{ds dt}{(s - t)^2}.$$

In order to examine the structure of this bilinear operator, we test it with L^2 basis functions $\phi_k(t) := e^{ikt}$, $k \in \mathbb{Z}$, obtaining $Q(\phi_k, \phi_k) = \mathcal{O}(|k|^3)$ and $Q(\phi_k, \phi_\ell) = 0$ for $k \neq \ell$.

By the fact that $Q(\phi_k, \phi_k)$ does not behave like k^3 but like $|k|^3$ the functional Q cannot be represented by an ordinary differential operator of third order. Instead we have to introduce *fractional powers of the Laplacian*. For $f : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}$ we define

$$J^s f := \sum_{k \in \mathbb{Z}} (1 + k^2)^{s/2} \hat{f}_k \phi_k$$

where $\hat{f}_k := \int_0^{2\pi} f(t) \phi_{-k}(t) dt$ is the k -th Fourier coefficient of f . Note that $J^2 f = f - \ddot{f}$. Now, using the L^2 inner product $\langle f, g \rangle_{L^2} := \int_0^{2\pi} \langle f(t), g(t) \rangle_{\mathbb{R}^3} dt = \sum_{k \in \mathbb{Z}} \hat{f}_k \overline{\hat{g}_k}$,

$$Q(\gamma, h) = c \langle J^2 \gamma, Jh \rangle_{L^2} + \text{lower order terms.} \quad (3)$$

The next step is isolating the highest-order derivative of δE . Still assuming arc-length parametrization ($|\dot{\gamma}| \equiv 1$) we arrive at

$$\delta E(\gamma, h) = 2Q(\gamma, h) + \text{l. o. t.} \stackrel{(3)}{=} c \langle J^2 \gamma, Jh \rangle_{L^2} + \text{l. o. t.} \quad (4)$$

which confirms the approximating property of Q . In order to write the lower order terms as $\langle \cdot, Jh \rangle_{L^2}$, we have to impose $h \perp \dot{\gamma}$, so let $h := P_{\dot{\gamma}^\perp} g := g - \langle g, \dot{\gamma} \rangle \dot{\gamma}$ be the projection of some arbitrary $g \in C^\infty$ onto the orthogonal complement of $\dot{\gamma}$. Furthermore, we need $\ddot{\gamma} \in L^3$. Now

$$\begin{aligned} \langle J^2 \gamma, Jh \rangle_{L^2} &= \langle \gamma - \ddot{\gamma}, J(P_{\dot{\gamma}^\perp} g) \rangle_{L^2} = - \langle \ddot{\gamma}, J(P_{\dot{\gamma}^\perp} g) \rangle_{L^2} + \text{l. o. t.} \\ &= - \langle \ddot{\gamma}, Jg \rangle_{L^2} + \langle \ddot{\gamma}, J(\langle g, \dot{\gamma} \rangle \dot{\gamma}) \rangle_{L^2} + \text{l. o. t.} \end{aligned} \quad (5)$$

In order to show that $\langle \ddot{\gamma}, J(\langle g, \dot{\gamma} \rangle \dot{\gamma}) \rangle_{L^2}$ is a lower order term, we introduce the matrix-valued function $f := \dot{\gamma} \dot{\gamma}^\top$ which gives $\langle g, \dot{\gamma} \rangle \dot{\gamma} = fg$. By arc-length parametrization we obtain $\ddot{\gamma} \perp \dot{\gamma}$ such that $f\ddot{\gamma} = 0$ and

$$\langle \ddot{\gamma}, J(\langle g, \dot{\gamma} \rangle \dot{\gamma}) \rangle_{L^2} = \langle \ddot{\gamma}, J(fg) \rangle_{L^2} = \langle fJ\ddot{\gamma}, g \rangle_{L^2} = - \langle J(f\ddot{\gamma}) - fJ\ddot{\gamma}, g \rangle_{L^2}. \quad (6)$$

The bilinear Fourier multiplier

$$\mathcal{M} : (f, \ddot{\gamma}) \mapsto J(f\ddot{\gamma}) - fJ\ddot{\gamma} = \sum_{k, \ell \in \mathbb{Z}} \underbrace{\frac{\sqrt{1+(k+\ell)^2} - \sqrt{1+\ell^2}}{\sqrt{1+k^2}}}_{\text{uniformly bounded}} \widehat{Jf}_k \hat{\gamma}_\ell \phi_{k+\ell} \quad (7)$$

produces a distribution, more precisely,

$$J^{-\frac{1}{2}-\varepsilon}\mathcal{M}(f, \ddot{\gamma}) \in L^2 \text{ for } \varepsilon > 0. \quad (8)$$

If γ is critical, the right-hand side of (4) vanishes. By (5) – (8), a critical point γ satisfies

$$\left\langle J^{\frac{1}{2}-\varepsilon}\ddot{\gamma}, J^{\frac{1}{2}+\varepsilon}g \right\rangle_{L^2} = \left\langle -J^{-\frac{1}{2}-\varepsilon}\mathcal{M}(f, \ddot{\gamma}) + \text{l. o. t.}, J^{\frac{1}{2}+\varepsilon}g \right\rangle_{L^2}$$

for all $g \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^3)$, so

$$J^{\frac{1}{2}-\varepsilon}\ddot{\gamma} = -J^{-\frac{1}{2}-\varepsilon}\mathcal{M}(f, \ddot{\gamma}) + \text{l. o. t.} \quad (9)$$

The right-hand side of (9) contains at most second order derivatives of γ and moreover belongs to L^2 , so $\gamma \in H^{2+\frac{1}{2}-\varepsilon}$, where $H^s := J^{-s}L^2$ is the *Sobolev space of fractional order* $s \in \mathbb{R}$.

This permits to establish a bootstrapping argument of the following scheme. If $\gamma \in H^s$, $s \geq 2$, the right-hand side of (9) belongs to H^{s-2} , which gives $\ddot{\gamma} \in H^{s-2+\frac{1}{2}-\varepsilon}$ and $\gamma \in H^{s+\frac{1}{2}-\varepsilon}$. This concludes the proof of Theorem 5.

Acknowledgements. This article contains an outline of a part of my Ph. D. thesis [10] at RWTH Aachen University. I would like to thank my advisor Prof. Dr. Heiko von der Mosel for his support.

REFERENCES

- [1] Simon Blatt, *The gradient flow of the Möbius energy near local minimizers*, preprint, 2009.
- [2] Simon Blatt and Philipp Reiter, *Does finite knot energy lead to differentiability?*, J. Knot Theory Ramifications, **17** (2008), 1281–1310.
- [3] Michael H. Freedman, Zheng-Xu He and Zhenghan Wang, *Möbius energy of knots and unknots*, Ann. of Math. (2), **139** (1994), 1–50.
- [4] Shinji Fukuhara, *Energy of a knot*, in “A Fête of Topology” (eds. Y. Matsumoto, T. Mizutani and S. Morita), Academic Press, (1988), 443–451.
- [5] Oscar Gonzalez, John H. Maddocks, Friedemann Schuricht and Heiko von der Mosel, *Global curvature and self-contact of nonlinearly elastic curves and rods*, Calc. Var. Partial Differential Equations, **14** (2002), 29–68.
- [6] Zheng-Xu He, *The Euler-Lagrange equation and heat flow for the Möbius energy*, Comm. Pure Appl. Math., **53** (2000), 399–431.
- [7] Robert B. Kusner and John M. Sullivan, *Möbius-invariant knot energies*, in “Ideal Knots” (eds. A. Stasiak, V. Katritch and L. H. Kauffman), World Scientific Publishing, (1998), 315–352.
- [8] Jun O’Hara, *Energy of a knot*, Topology, **30** (1991), 241–247.
- [9] Philipp Reiter. *All curves in a C^1 -neighbourhood of a given embedded curve are isotopic*, Report No. 4, Institut für Mathematik, RWTH Aachen, 2005.
- [10] Philipp Reiter, “Repulsive Knot Energies and Pseudodifferential Calculus. Rigorous Analysis and Regularity Theory for O’HARA’s Knot Energy Family $E^{(\alpha)}$, $\alpha \in [2, 3)$ ”, *Dissertation*, RWTH Aachen, 2009.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: reiter@instmath.rwth-aachen.de